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The Quantal Symmetries in the Composite Boson's System

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The composite Boson's system is quantizated in Faddeev-Senjanovic (FS) path integral quantizated formalism. The canonical Ward identities for proper vertices under local gauge transformation are derived. The fractional spins and fractional statistics are obtained by using the quantum Noether theorem.

KEY WORDS: path-integral quantization; ward identities; fractional spins; fractional statistics.

1. INTRODUCTION

In 1982, Klaus Von Klitzing and Dorda Pepper discovered the Fractional Quantum Hall Effective (FQHE) (Tsui *et al.*, 1992), which has been challenging the theory physicists. Laughlin's theory (Laughlin, 1983) is one of the most successful theories, which is able to explain the phenomenon. In Laughlin's theory, when the filling factor is odd, the fundamental states and quansi-particle states of the cooper pairing are fairly tallied with the results of the research in experiments (Clark *et al.*, 1988; Haldane, 1983; Simmons *et al.*, 1989). However, the theory is not completely satisfactory, because it fails to illuminate all the symmetries and does not provide a complete understanding of the problem (Khare, 2000). A complete understanding the phenomenon of superconductivity would not has been possible without the discovery of the Landau-Ginzburg mean field theory and the phenomenon of Cooper pairing. Therefore, this paper will discuss the composite Boson's system (Girvin and MacDonald, 1989; Lee and Zhang, 1991; Zhang *et al.*, 1989; Zhang *et al.*, 1989), which shows the phenomenon, FQHE. We firstly quantized the system in FS path-integral quantizated formalism. And we discussed the quantal symmetries by using the canonical Ward identities and the quantal Noether theorem.

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2. FS PATH-INTEGRAL QUANTIZATION

We consider the composite Boson's system; the action of the system is given by

$$I = \int d^3x \mathrm{L} \tag{1}$$

where

$$\mathrm{L} = \mathrm{L}_{CS} + \mathrm{L}_\phi \tag{2}$$

$$\mathrm{L}_{CS} = \frac{e\pi}{2\theta\Phi_0}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\rho \tag{3}$$

$$\begin{aligned}\mathrm{L}_\phi = {} & \phi^+(i\hbar\partial_t - eA_0 - ea_0)\phi - \frac{1}{2m}\left|(-i\hbar\nabla - \frac{e}{c}\vec{A} - \frac{e}{c}\vec{a})\phi\right|^2 \\ & -\frac{1}{2}\int d^3y\phi^+(x)\phi^+(y)V(x-y)\phi(x)\phi(y)\end{aligned} \tag{4}$$

where ϕ is Boson's fluid field, A_μ is electric field, a_μ is gauge field, Φ_0 is unit flux, θ is the parameter of Chern-Simons field.

According to FS path integral formalism, the canonical momentums conjugate to the fields ϕ, ϕ^+, A_μ are obtained

$$\pi_{\phi^+} = \frac{\partial \mathrm{L}}{\partial\dot{\phi}^+} = i\hbar\phi \tag{5}$$

$$\pi_\phi = \frac{\partial \mathrm{L}}{\partial\dot{\phi}} = i\hbar\phi^+ \tag{6}$$

$$\pi_i = \frac{\partial \mathrm{L}}{\partial\dot{a}_i} = \frac{e\pi}{2\theta\Phi_0}\varepsilon^{ji}a_j, \quad (i, j = 1, 2, 3) \tag{7}$$

$$\pi_0 = \frac{\partial \mathrm{L}}{\partial\dot{a}_0} = 0 \tag{8}$$

respectively. The primary constraints and canonical Hamiltonian density are given by

$$\Lambda^0 = \pi_0 \approx 0 \tag{9}$$

$$\begin{aligned}\mathrm{H}_c &= \pi_{\phi^+}\dot{\phi}^+ + \pi_\phi\dot{\phi} + \pi_\mu\dot{a}_\mu - \mathrm{L} \\ &= -\frac{e\pi}{2\theta\Phi_0}\varepsilon^{\mu i\rho}a_\mu\partial_i a_\rho + \frac{1}{2}\pi_{\phi^+}\dot{\phi}^+ + \frac{1}{2}\pi_\phi\dot{\phi} + \phi^+(eA_0 + ea_0)\phi + \frac{1}{2m}|(-i\hbar\nabla \\ &\quad -\frac{e}{c}\vec{A} - \frac{e}{c}\vec{a})\phi|^2 + \frac{1}{2}\int d^2\vec{r}_2\phi^+(\vec{r}_1)\phi^+(\vec{r}_2)V(\vec{r}_1 - \vec{r}_2)\phi(\vec{r}_1)\phi(\vec{r}_2)\end{aligned} \tag{10}$$

The total Hamiltonian density is written as

$$\mathrm{H}_T = \mathrm{H}_c + \lambda^0(t)\Lambda_0 \tag{11}$$

According to articles (Shizuya, 2002; Zhang and Ge, 2000), we can obtain the equations,

$$A_0 + a_0 = \nu_{eff} A_0 \tag{12}$$

$$\vec{A} + \vec{a} = \nu_{eff} \vec{A} \tag{13}$$

Then, the Eq. (8) can be written as

$$\begin{aligned}
\mathrm{H}_T = {}& \frac{1}{2}\pi_{\phi^+}\dot{\phi}^+ + \frac{1}{2}\pi_\phi \phi - \frac{e\pi}{2\theta\Phi_0}(\nu_{eff} - 1)^2 \varepsilon^{\mu i \rho} A_\mu \partial_i A_\rho \\
& + \phi^+(e\nu_{eff} A_0)\phi + \frac{1}{2m}|(-i\hbar\nabla - \frac{e}{c}\nu_{eff}\vec{A})\phi|^2 \\
& + \frac{1}{2}\int d^2\vec{r}_2 \phi^+(\vec{r}_1)\phi^+(\vec{r}_2)V(\vec{r}_1 - \vec{r}_2)\phi(\vec{r}_1)\phi(\vec{r}_2) + \lambda^0(t)\pi_0
\end{aligned} \tag{14}$$

The consistency condition $\{\Lambda^0, H_T\} \approx 0$ leads to secondary constraint

$$\Lambda^1 = \{\pi_0, \mathrm{H}_T\} = -\frac{\partial \mathrm{H}_T}{\partial A_0} = -\frac{e\pi}{\theta\Phi_0}(\nu_{eff} - 1)\varepsilon^{ij}\partial_i A_j - \phi^+\phi(e\nu_{eff}) \approx 0 \tag{15}$$

is first class constraint. The consistency condition $\{\Lambda^1, H_T\} \approx 0$ does not generate new constraints. It is easy to check that the constraints (Λ^0, Λ^1) are first class. According to the theory of canonical quantization of constrained Hamiltonian system, for each first-class constraints a corresponding gauge condition should be chosen. In the system, we choose Coulomb gauge as a corresponding gauge condition, for the constraint Λ^1. The other gauge constraint, for the constraint Λ^0, be obtained from the consistency condition of Coulomb gauge. The both gauge conditions are written as

$$\Omega_0 = \partial_i A_i \approx 0 \tag{16}$$

$$\Omega_1 = A_0 \approx 0 \tag{17}$$

In FS path integral quantization formalism, the phase-space generating functional of Green function for the system (2) was given by (Li and Jiang, 2002)

$$\begin{aligned}
Z[J, K, U] = {}& \int \mathrm{D}\varphi \mathrm{D}\pi \mathrm{D}\lambda \delta(\Lambda)\delta(\Omega)\det|\{\Lambda^l, \Omega_n\}| \\
& \times \exp\left\{ i\int d^3x \left(\mathrm{L}^P_{eff} + J\varphi + K\pi + U\lambda\right)\right\}
\end{aligned} \tag{18}$$

here we introduced exterior sources J, K and U with respect to the fields $\varphi = (\phi^+, \phi, A_\mu)$, $\pi = (\pi_{\phi^+}, \pi_\phi, \pi_\mu)$ and $\lambda = (\lambda^0, \lambda^1, \mu_1, \mu_2)$ respectively. It is easy to find out that det $|\{\Lambda^l, \Omega_n\}|$ is independent of field variables. Such, by using the properties of δ-function and integral of Grassman variables, the expression (15) is written as

$$Z[J, K, U] = \int \mathrm{D}\varphi \mathrm{D}\pi \mathrm{D}\lambda \exp\left\{i \int d^3x \big(\mathrm{L}^P_{eff} + J\varphi + K\pi + U\lambda\big)\right\} \tag{19}$$

with

$$\mathrm{L}^P_{eff} = \mathrm{L}^P + \mathrm{L}_m \tag{20}$$

$$\mathrm{L}^P = \pi_{\phi^+}\dot{\phi}^+ + \pi_\phi\dot{\phi} + \pi_\mu \dot{a}_\mu - \mathrm{H}_c \tag{21}$$

$$\mathrm{L}_m = \lambda^l \Lambda_l + \mu_n \Omega_n \tag{22}$$

3. CANONICAL WARD IDENTITIES

According to Dirac's conjecture, the gauge transformation of the system (2) can be constructed by the first class constraints (6) and (12)(Li and Jiang, 2002)

$$G = \int d^2x[\dot{\varepsilon}(x)\Lambda_0 - \varepsilon(x)\Lambda_1] = \int_V d^2x[\dot{\varepsilon}(x)\pi_0 - \varepsilon(x)e\nu_{eff}\phi^+\phi] \tag{23}$$

This generator produces the following transformations

$$\begin{cases} \delta A_0 = \{A_0, G\} = \dot{\varepsilon}(x) \\ \delta\pi_\phi = \{\pi_\phi, G\} = -\varepsilon(x)e\nu_{eff}\phi^+ \\ \delta\pi_{\phi^+} = \{\pi_{\phi^+}, G\} = -\varepsilon(x)e\nu_{eff}\phi \end{cases} \tag{24}$$

the others subentries equal zero. Under the transformations (21), the canonical Lagrangian (18) is invariant, the change of the effective Lagrangian (17) can be obtained

$$\delta\mathrm{L}^P_{eff} = \mu^1\dot{\varepsilon}(x) \tag{25}$$

Under the transformations (21), the generating functional Green function is invariant, the following Ward identities can be gained

$$\left[\dot{\varepsilon}(x)\frac{\delta}{\delta U^4} + \dot{\varepsilon}(x)J_0 - \varepsilon(x)e\nu_{eff}K_\phi\frac{\delta}{\delta J_{\phi^+}} - \varepsilon(x)e\nu_{eff}K_{\phi^+}\frac{\delta}{\delta J_\phi}\right]Z[J, K, U] = 0 \tag{26}$$

The generating functional of Green function $Z[J, K, U]$, the generating functional of connecting Green function $W[J, K, U]$, and the generating functional of proper vertices Green function $\Gamma[J, K, U]$ satisfy the following relations

$$Z[J, K, U] = \exp\{iW[J, K, U]\} \tag{27}$$

$$\Gamma[J, K, U] = W[J, K, U] - \int d^3x(J\varphi + K\pi + U\lambda) \tag{28}$$

whereas,

$$\begin{cases} \frac{\delta W}{\delta U^4} = \mu_1 \\ \frac{\delta \Gamma}{\delta \mu^1} = -U^4 \end{cases} \qquad \begin{cases} \frac{\delta W}{\delta J_0} = A_0 \\ \frac{\delta \Gamma}{\delta A_0} = -J_0 \end{cases} \tag{29}$$

$$\begin{cases} \frac{\delta W}{\delta J_{\phi^+}} = \phi^+ \\ \frac{\delta \Gamma}{\delta \pi_{\phi^+}} = -K_{\phi^+} \end{cases} \qquad \begin{cases} \frac{\delta W}{\delta J_{\phi}} = \phi \\ \frac{\delta \Gamma}{\delta \pi_{\phi}} = -K_{\phi} \end{cases} \tag{30}$$

Submitted (26) into (23), there is

$$\partial_0\mu^1 - \partial_0\frac{\delta\Gamma}{\delta A_0} + e\nu_{eff}\phi^+\frac{\delta\Gamma}{\delta\pi_\phi} + e\nu_{eff}\phi\frac{\delta\Gamma}{\delta\pi_{\phi^+}} = 0 \tag{31}$$

We functionally differentiate (27) with the fields variables ϕ and ϕ^+, and set all fields variables (including multiplier μ^1) equal to zero, $\phi = \phi^+ = \mu^1 = 0$. We obtain

$$\begin{aligned} \partial_0\frac{\delta^3\Gamma[0]}{\delta A_0(x_1)\delta\phi^+(x_2)\delta\phi(x_3)} &= \frac{e\nu_{eff}}{i\hbar}\delta(x_1 - x_3)\frac{\delta^2\Gamma[0]}{\delta\phi^+(x_1)\delta\phi(x_2)} \\ &\quad + \frac{e\nu_{eff}}{i\hbar}\delta(x_1 - x_2)\frac{\delta^2\Gamma[0]}{\delta\phi(x_1)\delta\phi^+(x_3)} \end{aligned} \tag{32}$$

Similarly, differentiating (27) many times with respect to field variables and setting all fields equal to zero, one can obtain various Ward identities for proper vertices.

4. FRACTIONAL SPIN AND FRACTIONAL SPIN AND FRACTIONAL SPIN AND FRACTIONAL STATISTICS

We will study the quantal symmetries the system (2), basing on Noether theorem in quantal level. Now, if the effective canonical action $I^P_{eff} = \int d^2x \mathrm{L}^P_{eff}$ is invariant under the following global transformations in extended phase space

$$\begin{cases} x^{\mu'} = x^\mu + \Delta x^\mu = x^\mu + \varepsilon_\sigma\tau^{\mu\sigma}(x, \varphi, \pi) \\ \varphi'(x') = \varphi(x) + \Delta\varphi(x) = \varphi(x) + \varepsilon_\sigma\xi^\sigma(x, \varphi, \pi) \\ \pi'(x') = \pi(x) + \Delta\pi(x) = \pi(x) + \varepsilon_\sigma\eta^\sigma(x, \varphi, \pi) \end{cases} \tag{33}$$

with $\varepsilon_\sigma(\sigma = 1, 2, \ldots, r)$, and the Jacobian of the transformations (29) equals to unity. According to canonical Noether theorem in quantal level, there is conserved law

$$Q^\sigma = \int_V d^3x\big[\pi(\xi^\sigma - \varphi_{,k}\tau^{k\sigma}) - \mathrm{H}_{eff}\tau^{0\sigma}\big] = \text{const} \tag{34}$$

where H_{eff} is effective Hamiltonian density connected to effective Lagrangian density L^P_{eff}. Under the spatial rotation, in (x,y) plane, taking follow of (5) and (21), we can obtain the quantal conserved angular momentum for this system (2)

$$L = \int d^2x\varepsilon^{ij}\left\{[x_i\pi_{\phi^+}\partial_j\phi^+ + x_i\pi_\phi\partial_j\phi] + \frac{e\pi}{2\theta\Phi_0}(\nu_{eff} - 1)^2\frac{1}{2}S^{kl}_{ij}\varepsilon^m_l A_m A_k \right.$$
$$\left. + \frac{e\pi}{2\theta\Phi_0}(\nu_{eff} - 1)^2 x_i\varepsilon^m_l A_m\partial^l A_j\right\} \tag{35}$$

where $S^{kl}_{ij} = \delta^k_i\delta^l_j - \delta^k_j\delta^l_i$. On the right hand of Eq. (31), the second term equals to zero by calculating, the third term can be written as

$$\int d^2x\frac{e\pi}{2\theta\Phi_0}(\nu_{eff} - 1)^2 x_i\varepsilon^{j'}_{i'}A_{j'}\partial_j A_{j'} = \frac{n^2e\pi}{2\theta}(\nu_{eff} - 1)^2\Phi_0 \tag{36}$$

for (Zhang and Le, 2000)

$$-\int d^2x\varepsilon^{ij}\partial_i A_j = n\Phi_0\,(n = 0,\ 1,\ 2, \ldots) \tag{37}$$

The composite Bosons are visualized as Bosons carrying even number 2p of flux quanta of the CS field $a_\mu(x)$ by following articles (Lopez and Fradkin, 1991; Xie *et al.*, 1991). According to article (Lopez and Fradkin, 1991), the effective factor ν_{eff} defined as

$$\nu_{eff} = \pm\frac{1}{2p \pm 1},\ (p = 0,\ 1,\ 2, \ldots). \tag{38}$$

Then, the Eq. (32) can be written as

$$\int d^2x\frac{e\pi}{2\theta\Phi_0}(\nu_{eff} - 1)^2 x_i\varepsilon^{j'}_{i'}A_{j'}\partial_j A_{j'} = \frac{m^2e\pi}{2\theta(2p \pm 1)^2}\Phi_0, \tag{39}$$

with $m = 2np = 0, 2, 4, \ldots$. We denote the spin operator by S,

$$S = \frac{m^2e\pi}{2\theta(2p \pm 1)^2}\Phi_0, \tag{40}$$

and the one-particle (any on) state is denoted by $|1>_{any}$, which carry one unit of charge. Then, if one rotates the one-particle state with S, one is obtained as

$$e^{i\beta S}|1>_{any} = e^{i\beta\left(\frac{m^2\pi}{2\theta(2p\pm1)^2}\right)}|1>_{any} \tag{41}$$

where β is rotation parameter. The ergenvalue of spin operator S is the spin s. Thus we obtain the following equation about the spin s and the coefficent in CS term,

$$s = \frac{m^2\pi}{2\theta(2p \pm 1)^2} \tag{42}$$

If we take β as 2 π, for $\theta = \frac{m^2}{(2p\pm 1)^3}$, $(p = 0,\ 1,\ 2, \ldots)$, the one-particle state picks up a minus sign implying it is a fermionic, and these values of θ let the spin s take half-integer values. While, for $\theta = m^2\pi/2p(2p \pm 1)^2$, $(p = 0,\ 1,\ 2, \ldots)$, the one-particle state does not change, and hence it becomes bosonic, and these values of θ let the spin s take integer values. For the other values of θ, the state would become anionic, and the spin s is fractional. In this paper, the values of θ do not agree with the values of article (Wang, 2004). Especially in the ferminonic and Bosonic states, the values of θ are mostly confined. At special values of θ, the spin s take anionic we should study deeply, such as

$$\theta = \frac{\pi}{2p},\ \frac{n\pi}{2p},\ \frac{\pi}{(2p \pm 1)},\ \frac{n\pi}{(2p \pm 1)}, \ldots, \tag{43}$$

and so on. The relation between the values of (and the fractional quantal Hall effect would be clear. The quantal symmetries of composite fermions are worthwhile of rediscussing (Wang, 2004).

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